Algebraic Inequalities

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1 The Rearrangement Inequality and Its Consequences

Theorem 1 (The Rearrangement Inequality). *Let $a_1 \le a_2 \le \cdots \le a_n$ and $b_1 \le b_2 \le \cdots \le b_n$ be real numbers. For any permutation $(a'_1, a'_2, \ldots, a'_n)$ of $(a_1, a_2, \ldots, a_n)$, we have*

$$\begin{aligned} a_1b_1 + a_2b_2 + \cdots + a_nb_n &\ge a'_1b_1 + a'_2b_2 + \cdots + a'_nb_n \\ &\ge a_nb_1 + a_{n-1}b_2 + \cdots + a_1b_n, \end{aligned}$$

with equality if and only if $a_i = a_{i+1}$ for all i such that $b_i < b_{i+1}$. In particular, if $b_1 < b_2 < \ldots < b_n$, the equality takes place if $(a'_1, a'_2, \ldots, a'_n)$ is equal to $(a_1, a_2, \ldots, a_n)$ or $(a_n, a_{n-1}, \ldots, a_1)$, respectively.

Proof. We will prove only the first inequality. Suppose $a'_i > a'_{i+1}$, then

$$a'_ib_i + a'_{i+1}b_{i+1} - a'_{i+1}b_i - a'_ib_{i+1} = (a'_i - a'_{i+1})(b_i - b_{i+1}) \le 0.$$

Moreover, if $b_i < b_{i+1}$, we have

$$a'_ib_i + a'_{i+1}b_{i+1} - a'_{i+1}b_i - a'_ib_{i+1} < 0,$$

which proves the statement since swapping neighboring pairs like the one above we can convert $(a'_1, a'_2, \ldots, a'_n)$ into $(a_1, a_2, \ldots, a_n)$. □

Corollary 1. *Let $a_1, a_2, \ldots, a_n$ be real numbers and $(a'_1, a'_2, \ldots, a'_n)$ be a permutation of $(a_1, a_2, \ldots, a_n)$. Then*

$$a_1^2 + a_2^2 + \cdots + a_n^2 \ge a_1a'_1 + a_2a'_2 + \cdots + a_na'_n.$$

*with borrowings from the article K.Wu and Andy Liu "The rearrangement inequality."

Corollary 2. *Let $a_1, a_2, \ldots, a_n$ be positive numbers and $(a'_1, a'_2, \ldots, a'_n)$ be a permutation of $(a_1, a_2, \ldots, a_n)$. Then*

$$\frac{a'_1}{a_1} + \frac{a'_2}{a_2} + \cdots + \frac{a'_n}{a_n} \geq n.$$

For the two inequalities above it is hard to formulate a general rule when inequality becomes an equality. But there are cases, when it is possible. For example, if $(a'_1, a'_2, \ldots, a'_n)$ is cyclic, then the equality takes place iff $a_1 = a_2 = \ldots = a_n$.

Exercise 1. *Let $a_1, a_2, \ldots, a_n$ be real numbers. Then*

$$a_1^2 + a_2^2 + \cdots + a_n^2 \geq a_1a_2 + a_2a_3 + \cdots + a_na_1 \tag{1}$$

with the equality iff $a_1 = a_2 = \ldots = a_n$.

Exercise 2. *Let $a_1, a_2, \ldots, a_n$ be positive numbers. Then*

$$\frac{a_n}{a_1} + \frac{a_1}{a_2} + \cdots + \frac{a_{n-1}}{a_n} \geq n \tag{2}$$

with the equality iff $a_1 = a_2 = \ldots = a_n$.

Exercise 3. *Prove that for all non-negative a, b, c the following two inequality hold*

$$a^2b + b^2c + c^2a \leq a^3 + b^3 + c^3 \leq \frac{a^4}{b} + \frac{b^4}{c} + \frac{c^4}{a}.$$

Establish when these are equalities.

Simple as it sounds, the Rearrangement Inequality is a result of fundamental importance. We shall derive from it many familiar and useful inequalities.

Theorem 2 (The Arithmetic Mean Geometric Mean Inequality). *Let $x_1, x_2, \ldots, x_n$ be positive numbers. Then*

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \geq \sqrt[n]{x_1x_2\cdots x_n},$$

with equality if and only if $x_1 = x_2 = \cdots = x_n$.

Proof. Let $G = \sqrt[n]{x_1x_2\cdots x_n}$, $a_1 = \frac{x_1}{G}, a_2 = \frac{x_1x_2}{G^2}, \ldots, a_n = \frac{x_1x_2\cdots x_n}{G^n} = 1$. By Corollary 2,

$$n \leq \frac{a_1}{a_n} + \frac{a_2}{a_1} + \cdots + \frac{a_n}{a_{n-1}} = \frac{x_1}{G} + \frac{x_2}{G} + \cdots + \frac{x_n}{G},$$

which is equivalent to $\frac{x_1 + x_2 + \cdots + x_n}{n} \geq G$. As the permutation was cyclic, equality holds if and only if $a_1 = a_2 = \cdots = a_n$, or $x_1 = x_2 = \cdots = x_n$. □

Theorem 3 (The Geometric mean Harmonic Mean Inequality). *Let $x_1, x_2, \ldots, x_n$ be positive numbers. Then*

$$\sqrt[n]{x_1 x_2 \cdots x_n} \geq \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}},$$

with equality if and only if $x_1 = x_2 = \cdots = x_n$.

Proof. Let $G, a_1, a_2, \ldots, a_n$ be as in Example 5. By Corollary 2,

$$n \leq \frac{a_1}{a_2} + \frac{a_2}{a_3} + \cdots + \frac{a_n}{a_1} = \frac{G}{x_1} + \frac{G}{x_2} + \cdots + \frac{G}{x_n},$$

which is equivalent to

$$G \geq \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}}.$$

As the permutation was cyclic, equality holds if and only if $x_1 = x_2 = \cdots = x_n$. □

Theorem 4 (The Root Mean Square Arithmetic Mean Inequality). *Let $x_1, x_2, \ldots, x_n$ be real numbers. Then*

$$\sqrt{\frac{x_1^2 + x_2^2 + \cdots + x_n^2}{n}} \geq \frac{x_1 + x_2 + \cdots + x_n}{n},$$

with equality if and only if $x_1 = x_2 = \cdots = x_n$.

Proof. By Corollary 1, we have

$$\begin{array}{rcl} x_1^2 + x_2^2 + \cdots + x_n^2 & \geq & x_1 x_2 + x_2 x_3 + \cdots + x_n x_1, \\ x_1^2 + x_2^2 + \cdots + x_n^2 & \geq & x_1 x_3 + x_2 x_4 + \cdots + x_n x_2, \\ \cdots & \geq & \cdots \\ x_1^2 + x_2^2 + \cdots + x_n^2 & \geq & x_1 x_n + x_2 x_1 + \cdots + x_n x_{n-1}. \end{array}$$

Adding these and $x_1^2 + x_2^2 + \cdots + x_n^2 = x_1^2 + x_2^2 + \cdots + x_n^2$, we have

$$n(x_1^2 + x_2^2 + \cdots + x_n^2) \geq (x_1 + x_2 + \cdots + x_n^2)^2,$$

which is equivalent to the desired result. Equality holds if and only if $x_1 = x_2 = \cdots = x_n$. □

Exercise 4. *Prove that for positive a, b, c*

$$a^a b^b c^c \geq a^b b^c c^a.$$

When does this become an equality?

2 Cauchy's Inequality and Its Consequences

The full name of this inequality is Cauchy-Schwarz-Bunyakovski Inequality.

Theorem 5 (Cauchy's Inequality). *Let $a_1, a_2, \ldots a_n, b_1, b_2, \ldots, b_n$ be real numbers. Then*

$$(a_1b_1 + a_2b_2 + \cdots + a_nb_n)^2 \le (a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + b_n^2),$$

with equality if and only if for some constant $k, a_i = kb_i$ for $1 \le i \le n$ or $b_i = ka_i$ for $1 \le i \le n$.

Proof. If $a_1 = a_2 = \cdots = a_n = 0$ or $b_1 = b_2 = \cdots = b_n = 0$, the result is trivial. Otherwise, define $S = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$ and $T = \sqrt{b_1^2 + b_2^2 + \cdots + b_n^2}$. Since both are non-zero, we may let $x_i = \dfrac{a_i}{S}$ and $x_{n+i} = \dfrac{b_i}{T}$ for $1 \le i \le n$. By Corollary 1,

$$\begin{aligned} 2 &= \frac{a_1^2 + a_2^2 + \cdots + a_n^2}{S^2} + \frac{b_1^2 + b_2^2 + \cdots + b_n^2}{T^2} \\ &= x_1^2 + x_2^2 + \cdots + x_{2n}^2 \\ &\ge x_1x_{n+1} + x_2x_{n+2} + \cdots + x_nx_{2n} + x_{n+1}x_1 + x_{n+2}x_2 + \cdots + x_{2n}x_n \\ &= \frac{2(a_1b_1 + a_2b_2 + \cdots + a_nb_n)}{ST}, \end{aligned}$$

which is equivalent to the desired result. Equality holds if and only if $x_i = x_{n+i}$ for $1 \le i \le n$, or $a_iT = b_iS$ for $1 \le i \le n$. $\square$

Example 1. *For positive real numbers $x_1, \ldots, x_n$ we have*

$$(x_1^2 + x_2^2 + \ldots + x_n^2)(1^2 + 1^2 + \ldots + 1^2) \ge (x_1 + x_2 + \ldots + x_n)^2,$$

from which

$$\frac{x_1^2 + x_2^2 + \ldots + x_n^2}{n} \ge \left(\frac{x_1 + x_2 + \ldots + x_n}{n}\right)^2$$

and we again obtain the Root Mean Square Arithmetic Mean Inequality.

Example 2. *Prove the inequality*

$$\sin\alpha\,\sin\beta + \cos\alpha + \cos\beta \le 2.$$

Indeed,

$$\sin\alpha\,\sin\beta + \cos\alpha\cdot 1 + 1\cdot\cos\beta \le \sqrt{\sin^2\alpha + \cos^2\alpha + 1^2}\sqrt{\sin^2\beta + 1^2 + \cos^2\beta} = 2.$$

Exercise 5. *Use Cauchy's inequality to prove that, for all positive $x_1, \ldots, x_n$,*

$$(x_1 + x_2 + \ldots + x_n)\left(\frac{1}{x_1} + \frac{1}{x_2} + \ldots + \frac{1}{x_n}\right) \geq n^2,$$

and show that this is equivalent to Arithmetic Mean Harmonic Mean Inequality.

Another very useful form of Cauchy's inequality is as follows.

Theorem 6. *Let $x_1, \ldots, x_n$ be arbitrary real numbers and $y_1, \ldots, y_n$ be positive real numbers. Then*

$$\frac{x_1^2}{y_1} + \frac{x_2^2}{y_2} + \ldots + \frac{x_n^2}{y_n} \geq \frac{(x_1 + x_2 + \ldots + x_n)^2}{y_1 + y_2 + \ldots + y_n}.$$

Solution. One can reduce this to the Cauchy's inequality by substituting $a_i = \frac{x_i}{\sqrt{y_i}}$ and $b_i = \sqrt{y_i}$. $\square$

Let us reformulate the Cauchy's inequality as follows:

Theorem 7.

$$\sum_{k=1}^{n} x_k^2 = \max\left\{\left(\sum_{k=1}^{n} x_k y_k\right)^2 \left(\sum_{k=1}^{n} y_k^2\right)^{-1}\right\}, \tag{3}$$

where $(y_1, \ldots, y_n) \neq (0, \ldots, 0)$ and arbitrary otherwise.

Proof. Obvious from the original Cauchy's inequality. Indeed it implies that

$$\sum_{k=1}^{n} x_k^2 \geq \left(\sum_{k=1}^{n} x_k y_k\right)^2 \left(\sum_{k=1}^{n} y_k^2\right)^{-1}.$$

On the other hand both sides are equal, for example, at $x_k = y_k$. $\square$

One obvious but very useful corollary.

Corollary 3.

$$\sum_{k=1}^{n} x_k^2 = \max\left(\sum_{k=1}^{n} x_k y_k\right)^2, \tag{4}$$

where $\sum_{k=1}^{n} y_k = 1$ and arbitrary otherwise.

Firstly we will exploit the theorem.

Example 3 (IMO 95). *Let a, b, c be positive real numbers such that $abc = 1$. Prove that*

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2}.$$

Solution. The idea is to "dismantle" the complicated expression on the left. We will represent all three summands on the left as squares:

$$x_1^2 = \frac{1}{a^3(b+c)}, \quad x_2^2 = \frac{1}{b^3(c+a)}, \quad x_3^2 = \frac{1}{c^3(a+b)},$$

and will choose y_1, y_2, y_3 as follows:

$$y_1^2 = a(b+c), \quad y_2^2 = b(c+a), \quad y_3 = c(a+b).$$

The idea is clear: to make both terms on the right of (4) managable. As $abc = 1$, we get

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \geq$$

$$\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)^2 (a(b+c) + b(c+a) + c(a+b))^{-1} = \frac{ab+bc+ca}{2} \geq \frac{3}{2},$$

since AM-GM implies

$$ab + bc + ca \geq 3\sqrt[3]{a^2b^2c^2} = 3.$$

The inequality is proven. □

Exercise 6 (Kvant, 1985). *Prove that for any positive numbers a, b, c, d*

$$\frac{a}{b+c} + \frac{b}{c+d} + \frac{c}{d+a} + \frac{d}{a+b} \geq 2.$$

Example 4 (GDR 67). *Prove that, if $n \geq 2$, and $a_1, \ldots, a_n$ are positive numbers whose sum is S, then*

$$\frac{a_1}{S-a_1} + \frac{a_2}{S-a_2} + \ldots + \frac{a_n}{S-a_n} \geq \frac{n}{n-1}.$$

Solution. We will apply the theorem for the numbers

$$x_k^2 = \frac{a_k}{S-a_k}, \qquad y_k^2 = a_k(S-a_k).$$

We will obtain

$$\sum_{k=1}^{n} \frac{a_k}{S-a_k} \geq \left(\sum_{k=1}^{n} a_k\right)^2 \cdot \left(\sum_{k=1}^{n} a_k(S-a_k)\right)^{-1} = S^2 \left(\sum_{k=1}^{n} a_k(S-a_k)\right)^{-1}.$$

Hence it is enough to show that

$$S^2\left(\sum_{k=1}^{n} a_k(S-a_k)\right)^{-1} \geq \frac{n}{n-1}.$$

or $(n-1)S^2 \geq n\sum_{k=1}^{n} a_k(S-a_k) = nS^2 - n\sum_{k=1}^{n} a_k^2$, which follow from the RootMean Square - Arithmetic Mean inequality. □

Exercise 7 (Mongolia, 1996). *Prove that for any positive numbers a, b, c, d satisfying $a+b+c+d = 1$*

$$\frac{1}{1-\sqrt{a}} + \frac{1}{1-\sqrt{b}} + \frac{1}{1-\sqrt{c}} + \frac{1}{1-\sqrt{d}} \geq 8.$$

Let us now see what the corollary has to offer.

Example 5 (Beckenbach Inequality). *For positive numbers $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ the following inequality holds:*

$$\frac{\sum_{k=1}^{n}(x_k+y_k)^2}{\sum_{k=1}^{n} x_k + \sum_{k=1}^{n} y_k} \leq \frac{\sum_{k=1}^{n} x_k^2}{\sum_{k=1}^{n} x_k} + \frac{\sum_{k=1}^{n} y_k^2}{\sum_{k=1}^{n} y_k}.$$

Solution. Let $X = x_1 + \ldots + x_n$ and $Y = y_1 + \ldots + y_n$. We claim that it is sufficient to show that for any positive numbers $z_1, \ldots, z_n$

$$\frac{\left(\sum_{k=1}^{n}(x_k+y_k)z_k\right)^2}{X+Y} \leq \frac{\left(\sum_{k=1}^{n} x_k z_k\right)^2}{X} + \frac{\left(\sum_{k=1}^{n} y_k z_k\right)^2}{Y} \tag{5}$$

Indeed, if (5) was true, then due to Corollary 4

$$\frac{\sum_{k=1}^{n}(x_k+y_k)^2}{\sum_{k=1}^{n} x_k + \sum_{k=1}^{n} y_k} = \frac{\max\left(\sum_{k=1}^{n}(x_k+y_k)z_k\right)^2}{X+Y} = \frac{\left(\sum_{k=1}^{n}(x_k+y_k)z_k^{'}\right)^2}{X+Y} \leq$$

$$\frac{\left(\sum_{k=1}^{n} x_k z_k^{'}\right)^2}{X} + \frac{\left(\sum_{k=1}^{n} y_k z_k^{'}\right)^2}{Y} \leq \frac{\max\left(\sum_{k=1}^{n} x_k z_k\right)^2}{X} + \frac{\max\left(\sum_{k=1}^{n} y_k z_k\right)^2}{Y} =$$

$$\frac{\sum_{k=1}^{n} x_k^2}{\sum_{k=1}^{n} x_k} + \frac{\sum_{k=1}^{n} y_k^2}{\sum_{k=1}^{n} y_k}.$$

□

To prove (5) we denote

$$x = \sum_{k=1}^{n} \frac{x_k z_k}{\sqrt{X}}, \qquad y = \sum_{k=1}^{n} \frac{y_k z_k}{\sqrt{Y}}.$$

Then (5) becomes

$$\frac{(x\sqrt{X}+y\sqrt{Y})^2}{X+Y} \le x^2+y^2,$$

which is easy to prove by expanding of brackets.

Exercise 8 (Minkowski's inequality). *Using the same idea prove that for any positive integers* $x_1,\dots,x_n$ *and* $y_1,\dots,y_n$

$$\sqrt{x_1^2+x_2^2+\ldots+x_n^2}+\sqrt{y_1^2+y_2^2+\ldots+y_n^2} \ge \sqrt{(x_1+y_1)^2+(x_2+y_2)^2+\ldots+(x_n+y_n)^2}.$$

The trick used in the last two problems and formulated in Corollary is called quasilinearisation of Cauchy's inequality. You can do it with some other inequalities too. The quasilinearisation of AM-GM will look as follows.

Theorem 8. *For any positive integers* $x_1,\dots,x_n$

$$\sqrt[n]{x_1x_2\cdots x_n} = \min \sum_{k=1}^{n} \frac{x_k y_k}{n},$$

where $y_1,\dots,y_n$ *are positive numbers such that* $y_1y_2\cdots y_n = 1$.

Exercise 9. *Prove this theorem.*

Exercise 10. *Use the theorem proved in the previous exercise and prove another Minkowski's inequality*

$$\sqrt[n]{x_1x_2\cdots x_n}+\sqrt[n]{y_1y_2\cdots y_n} \le \sqrt[n]{(x_1+y_1)(x_2+y_2)\cdots(x_n+y_n)}.$$

3 Chebyshev's Inequality

This is also known as Tchebychef's inequality.

Definition. Let $a_1,a_2,\dots,a_n$ and $b_1,b_2,\dots,b_n$ be two monotonic (monotone increasing or monotone decreasing) sequences of real numbers. These sequences are called *similarly directed*, if

$$(a_i-a_j)(b_i-b_j) \ge 0$$

for all $i \le j$. The sequences are called *oppositely directed*, if

$$(a_i-a_j)(b_i-b_j) \le 0$$

for all $i \le j$.

Theorem 9 (Chebyshev's Inequality). *Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be two monotonic sequences of real numbers. Then*

$$\frac{a_1 + a_2 + \ldots + a_n}{n} \cdot \frac{b_1 + b_2 + \ldots + b_n}{n} \le \frac{a_1 b_1 + a_2 b_2 + \ldots + a_n b_n}{n}, \tag{6}$$

if the two sequences are similarly directed and

$$\frac{a_1 + a_2 + \ldots + a_n}{n} \cdot \frac{b_1 + b_2 + \ldots + b_n}{n} \ge \frac{a_1 b_1 + a_2 b_2 + \ldots + a_n b_n}{n}, \tag{7}$$

if they are oppositely directed.

Proof. Suppose the two sequences are similarly directed. By the Rearrangement Inequality

$$\begin{array}{rcl} a_1 b_1 + a_2 b_2 + \ldots + a_n b_n & = & a_1 b_1 + a_2 b_2 + \ldots + a_n b_n, \\ a_1 b_1 + a_2 b_2 + \ldots + a_n b_n & \ge & a_1 b_2 + a_2 b_3 + \ldots + a_n b_1, \\ \cdots & \ge & \cdots \\ a_1 b_1 + a_2 b_2 + \ldots + a_n b_n & \ge & a_1 b_n + a_2 b_1 + \ldots + a_n b_{n-1}. \end{array}$$

Adding all of them together we obtain

$$n(a_1 b_1 + a_2 b_2 + \ldots + a_n b_n) \ge (a_1 + a_2 + \ldots + a_n)(b_1 + b_2 + \ldots + b_n),$$

which is equivalent to Chebyshev's Inequality (6). The second Chebyshev's Inequality (6) can be proved similarly. □

Exercise 11. *Prove Chebyshev's Inequality (7).*

Example 6. *Prove that for all real numbers $x_1, \ldots, x_n$*

$$(x_1 + x_2 + \ldots + x_n)\left(x_1^3 + x_2^3 + \ldots + x_n^3\right) \le n\left(x_1^4 + x_2^4 + \ldots + x_n^4\right).$$

Indeed, we may assume that $x_1 \le x_2 \le \ldots \le x_n$. But then $x_1^3 \le x_2^3 \le \ldots \le x_n^3$ and Chebyshev's Inequality (6) is applicable and gives the result.

Exercise 12. *Prove that for all real numbers $x_1, \ldots, x_n$*

$$\left(x_1^5 + x_2^5 + \ldots + x_n^5\right)\left(\frac{1}{x_1^2} + \frac{1}{x_2^2} + \ldots + \frac{1}{x_n^2}\right) \ge n\left(x_1^3 + x_2^3 + \ldots + x_n^3\right).$$

4 Jensen's Inequality

This method might require a little bit of Calculus.

Definition 1. *Let us recall that a function $f(x)$ on a segment $[a, b]$ is* concave up *if for all $x_1, x_2 \in [a, b]$*

$$f\left(\frac{x_1 + x_2}{2}\right) \le \frac{f(x_1) + f(x_2)}{2}, \tag{8}$$

and concave down *if for all $x_1, x_2 \in [a, b]$*

$$f\left(\frac{x_1 + x_2}{2}\right) \ge \frac{f(x_1) + f(x_2)}{2}. \tag{9}$$

Proposition 1. *If the function $f(x)$ is continuous on $[a, b]$ and twice differentiable on (a, b), then it is concave up iff $f''(x) \ge 0$ and it is concave down iff $f''(x) \le 0$.*

Theorem 10 (Jensen's Inequality). *Let $n \ge 2$ and $\alpha_1, \ldots, \alpha_n$ be nonnegative real numbers such that $\alpha_1 + \ldots + \alpha_n = 1$. Then*

1. *For an arbitrary concave up function $f(x)$ on a segment $[a, b]$*

$$f\left(\alpha_1 x_1 + \cdots + \alpha_n x_n\right) \le \alpha_1 f(x_1) + \cdots + \alpha_n f(x_n)$$

   for all $x_1, \ldots, x_n \in [a, b]$.

2. *For an arbitrary concave down function $f(x)$ on a segment $[a, b]$*

$$f\left(\alpha_1 x_1 + \cdots + \alpha_n x_n\right) \ge \alpha_1 f(x_1) + \cdots + \alpha_n f(x_n)$$

   for all $x_1, \ldots, x_n \in [a, b]$.

Example 7. *Prove that for all nonnegative $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$,*

$$\left(x_1^3 + x_2^3 \ldots + x_n^3\right)\left(y_1^3 + y_2^3 \ldots + y_n^3\right)^2 \ge \left(x_1 y_1^2 + x_2 y_2^2 \ldots + x_n y_n^2\right)^3.$$

To prove this, let us denote $S = y_1^3 + \ldots + y_n^3$. Then our inequality can be rewritten as

$$\frac{x_1^3 + x_2^3 \ldots + x_n^3}{S} \ge \frac{\left(x_1 y_1^2 + x_2 y_2^2 \ldots + x_n y_n^2\right)^3}{S^3},$$

or

$$\frac{y_1^3}{S}\left(\frac{x_1}{y_1}\right)^3 + \frac{y_2^3}{S}\left(\frac{x_2}{y_2}\right)^3 + \ldots + \frac{y_n^3}{S}\left(\frac{x_n}{y_n}\right)^3 \ge \left(\frac{y_1^3}{S}\left(\frac{x_1}{y_1}\right) + \frac{y_2^3}{S}\left(\frac{x_2}{y_2}\right) + \ldots + \frac{y_n^3}{S}\left(\frac{x_n}{y_n}\right)\right)^3,$$

which follows from concavity up of the function $f(x) = x^3$ for nonnegative x. Indeed, $f''(x) = 6x \ge 0$ for $x \ge 0$ and

$$\frac{y_1^3}{S} + \frac{y_2^3}{S} + \ldots + \frac{y_n^3}{S} = \frac{S}{S} = 1.$$

Often you need to take logs before applying Jensen's inequality.

Example 8. *Prove that*

$$\left(1+\frac{1}{x}\right)\left(1+\frac{1}{y}\right)\left(1+\frac{1}{z}\right) \geq 64,$$

where $x, y, z > 0$ and $x+y+z=1$.

After taking logs on both sides, we get

$$\log_2\left(1+\frac{1}{x}\right)+\log_2\left(1+\frac{1}{y}\right)+\log_2\left(1+\frac{1}{z}\right) \geq 6.$$

We check that the function $f(t)=\log_2(1+1/t)$ is concave up. Indeed, calculation show that

$$f''(t)=\frac{1}{t^2}-\frac{1}{(t+1)^2}>0.$$

Hence by Jensen's inequality

$$\frac{1}{3}\log_2\left(1+\frac{1}{x}\right)+\frac{1}{3}\log_2\left(1+\frac{1}{y}\right)+\frac{1}{3}\log_2\left(1+\frac{1}{z}\right) \geq \log_2\left(1+\frac{3}{x+y+z}\right)=2$$

and the desired inequality follows.

Exercise 13. *Using Jensen's inequality prove that*

$$a^{a/a+b+c}\cdot b^{b/a+b+c}\cdot c^{c/a+b+c} \leq \frac{a^2+b^2+c^2}{a+b+c},$$

where a, b, c are positive integers.

One of the powerful tools is the so-called weighted AM-GM inequality.

Theorem 11 (Weighted AM-GM inequality). *Let $w_1, \ldots, w_n$ be non-negative reals such that $w_1+\ldots+w_n=1$, then for any non-negative reals $x_1, \ldots, x_n$*

$$w_1x_1+w_2x_2+\ldots+w_nx_n \geq x_1^{w_1}x_2^{w_2}\ldots x_n^{w_n}$$

with equality if and only if all the x_i with $w_i \neq 0$ are equal.

Exercise 14. *Deduce Weighted AM-GM inequality from Jensen's inequality.*

Weighted AM-GM inequality is quite powerful but sometimes difficult to use. We illustrate this on the following example.

Example 9. *Prove that for all positive reals a, b, c, d*

$$a^4b + b^4c + c^4d + d^4a \geq abcd(a + b + c + d).$$

Solution. By weighted AM-GM

$$\frac{23a^4b + 7b^4c + 11c^4d + 10d^4a}{51} \geq \sqrt[51]{a^{102}b^{51}c^{51}d^{51}} = a^2bcd.$$

It remains to symmetrise this inequality.

How can we find weights $w_1 = \frac{23}{51}$, $w_2 = \frac{7}{51}$, $w_3 = \frac{11}{51}$, $w_4 = \frac{10}{51}$? Aiming to get in this particular case

$$(a^4b)^{w_1}(b^4c)^{w_2}(c^4d)^{w_3}(d^4a)^{w_4} = a^2bcd,$$

we solving the system of linear equations

$$\begin{aligned} w_1 + w_2 + w_3 + w_4 &= 1, \\ 4w_1 + w_4 &= 2, \\ 4w_2 + w_1 &= 1, \\ 4w_3 + w_2 &= 1. \end{aligned}$$

and find the weights. $\square$

Weighted AM-GM plus symmetrisation is a powerful tool.

Exercise 15. *Prove that if $abc > 0$, then*

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \leq \frac{a^8 + b^8 + c^8}{a^3b^3c^3}.$$

by reducing this inequality to weighted AM-GM inequality.

5 Schur's Inequality

The following simple inequality can be often useful.

Theorem 12. *Let $t \geq 1$ be a real number. Then for all non-negative real numbers x, y, z*

$$x^r(x - y)(x - z) + y^r(y - z)(y - x) + z^r(z - x)(z - y) \geq 0,$$

where the equality occurs only when $x = y = z \neq 0$ or some two of x, y, z are equal and the third is zero.

Proof. Due to the symmetry of the inequality we may assume that $x \geq y \geq z$. Then our inequality can be rewritten as

$$(x-y)\left[x^r(x-z)-y^r(y-z)\right]+z^r(x-z)(y-z) \geq 0,$$

which is obvious since every term of it is non-negative. $\square$

Exercise 16. *Prove that for any positive a, b, c*

$$a^3+b^3+c^3+3abc \geq a^2(b+c)+b^2(a+c)+c^2(a+b). \tag{10}$$

6 Muirhead's Inequality

Let us consider the set S_n of sequences $(\alpha) = (\alpha_1, \ldots, \alpha_n)$ with the following two properties:

$$\alpha_1+\alpha_2+\ldots+\alpha_n = 1, \tag{11}$$

$$\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_n \geq 0. \tag{12}$$

For any two sequences (α) and (β) from S we say that (α) *majorises* (β) if

$$\alpha_1+\alpha_2+\ldots+\alpha_r \geq \beta_1+\beta_2+\ldots+\beta_r$$

for all $1 \leq r < n$. We denote this as $(\alpha) \succeq (\beta)$. If $(\alpha) \succeq (\beta)$ and $(\alpha) \neq (\beta)$, we will write $(\alpha) \succ (\beta)$.

Example 10. $(1,0,0) \succ (\frac{1}{2}, \frac{1}{2}, 0) \succ (\frac{1}{2}, \frac{1}{3}, \frac{1}{6}) \succ (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

Now we introduce one more notation: for (α) from S_n we denote

$$x^{(\alpha)} = \frac{1}{n!}(x_1^{\alpha_1}x_2^{\alpha_2}\ldots x_n^{\alpha_n}+\ldots),$$

where the dots denote all $n!-1$ terms obtained by permutations of α's.

Example 11. *Let $n=3$. Then*

$$x^{(1,0,0)} = \frac{1}{6}(x_1^1x_2^0x_3^0+x_1^1x_3^0x_2^0+x_2^1x_1^0x_3^0+x_2^1x_3^0x_1^0+x_3^1x_1^0x_2^0+x_3^1x_2^0x_1^0) = \frac{1}{3}(x_1+x_2+x_3);$$

$$x^{(\frac{1}{2},\frac{1}{2},0)} = \frac{1}{6}(x_1^{\frac{1}{2}}x_2^{\frac{1}{2}}x_3^0+x_2^{\frac{1}{2}}x_1^{\frac{1}{2}}x_3^0+x_1^{\frac{1}{2}}x_3^{\frac{1}{2}}x_2^0+x_3^{\frac{1}{2}}x_1^{\frac{1}{2}}x_2^0+x_2^{\frac{1}{2}}x_3^{\frac{1}{2}}x_1^0+x_3^{\frac{1}{2}}x_2^{\frac{1}{2}}x_1^0) =$$

$$\frac{1}{3}(\sqrt{x_1x_2}+\sqrt{x_1x_3}+\sqrt{x_2x_3});$$

$$x^{(\frac{1}{3},\frac{1}{3},\frac{1}{3})} = \sqrt[3]{x_1x_2x_3}.$$

Theorem 13 (Muirhead). *If $(\alpha) \succeq (\beta)$, then the inequality*

$$x^{(\alpha)} \geq x^{(\beta)}$$

holds for all non-negative $x_1, \dots, x_n$. There is equality only when $(\alpha) = (\beta)$ or all the x_i are equal.

Proof. We will prove this theorem for the case $n = 3$. The general case can be proved similarly. We assume that $(\alpha) \neq (\beta)$ and not all the x_i are equal. Let us consider the following three partial cases from which the general case will follow.

(a) Let $(\alpha) = (\alpha_1, \alpha_2, \alpha_3)$, and ρ be a positive real number such that $\rho < \alpha_1 - \alpha_2$ and $(\alpha') = (\alpha_1 - \rho, \alpha_2 + \rho, \alpha_3)$. Then

$$3!\left(x^{(\alpha)} - x^{(\alpha')}\right) = (x_1^{\alpha_1}x_2^{\alpha_2} + x_1^{\alpha_2}x_2^{\alpha_1} - x_1^{\alpha'_1}x_2^{\alpha'_2} - x_1^{\alpha'_2}x_2^{\alpha'_1})x_3^{\alpha_3} +$$

$$(x_1^{\alpha_1}x_3^{\alpha_2} + x_1^{\alpha_2}x_3^{\alpha_1} - x_1^{\alpha'_1}x_3^{\alpha'_2} - x_1^{\alpha'_2}x_3^{\alpha'_1})x_2^{\alpha_3} +$$
$$(x_2^{\alpha_1}x_3^{\alpha_2} + x_2^{\alpha_2}x_3^{\alpha_1} - x_2^{\alpha'_1}x_3^{\alpha'_2} - x_2^{\alpha'_2}x_3^{\alpha'_1})x_1^{\alpha_3}.$$

The latter expression is positive. Indeed, if $x_i \neq x_j$, then

$$x_i^{\alpha_1}x_j^{\alpha_2} + x_i^{\alpha_2}x_j^{\alpha_1} - x_i^{\alpha'_1}x_j^{\alpha'_2} - x_i^{\alpha'_2}x_j^{\alpha'_1} =$$

$$(x_ix_j)^{\alpha_2}\left[(x_i^{\alpha_1-\alpha_2-\rho} - x_j^{\alpha_1-\alpha_2-\rho})(x_i^{\rho} - x_j^{\rho})\right] > 0,$$

since both round brackets are either both positive or both negative. Since not all the x_i are equal, there will be $x_1 \neq x_2$, or $x_1 \neq x_3$, or $x_2 \neq x_3$. This secures that $x^{(\alpha)} > x^{(\alpha')}$.

(b) Similarly, if $\rho < \alpha_2 - \alpha_3$ and $(\alpha'') = (\alpha_1, \alpha_2 - \rho, \alpha_3 + \rho)$, then $x^{(\alpha)} > x^{(\alpha'')}$.

(c) The same argument also shows that if $\alpha_2 = \alpha'_2$, then, for any $\rho \leq \alpha_1 - \alpha_3$, and $(\alpha''') = (\alpha_1 - \rho, \alpha_2, \alpha_3 + \rho)$, then $x^{(\alpha)} > x^{(\alpha''')}$.

Suppose now that $\alpha_2 < \beta_2$. Then $\alpha_2 < \beta_1$, and $\rho = \alpha_1 - \beta_1 < \alpha_1 - \alpha_2$. Then by the case (a), $x^{(\alpha)} > x^{(\alpha')}$, where $(\alpha') = (\alpha_1 - \rho, \alpha_2 + \rho, \alpha_3) = (\beta_1, \alpha_2 + \rho, \alpha_3)$. As $\alpha_3 < \beta_3$, $\alpha_2 + \rho > \beta_2$, hence $x^{(\alpha')} > x^{(\beta)}$ by (b). Thus $x^{(\alpha)} > x^{(\beta)}$, as required. The case $\alpha_2 > \beta_2$ is considered similarly using the cases (b) and (a). If $\alpha_2 = \beta_2$, then the statement follows straight from (c). □

Example 12. *Prove that for all non-negative a, b, c*

$$\frac{1}{3}(a^2 + b^2 + c^2) \geq \frac{1}{3}(ab + ac + bc) \geq \sqrt[3]{a^2b^2c^2}.$$

Solution. For all non-negative x_1, x_2, x_3 by Muirheads theorem

$$x^{(1,0,0)} \geq x^{(\frac{1}{2},\frac{1}{2},0)} \geq x^{(\frac{1}{3},\frac{1}{3},\frac{1}{3})}.$$

Hence

$$\frac{1}{3}(x_1 + x_2 + x_3) \geq \frac{1}{3}(\sqrt{x_1x_2} + \sqrt{x_1x_3} + \sqrt{x_2x_3}) \geq \sqrt[3]{x_1x_2x_3}$$

or, after the substitution $a = \sqrt{x_1}$, $b = \sqrt{x_2}$, $c = \sqrt{x_3}$,

$$\frac{1}{3}(a^2 + b^2 + c^2) \geq \frac{1}{3}(ab + ac + bc) \geq \sqrt[3]{a^2b^2c^2}.$$

□

Exercise 17. *Prove that for all non-negative a, b, c the following two inequalities hold*

$$2(a^3 + b^3 + c^3) \geq a^2(b + c) + b^2(c + a) + c^2(a + b) \geq 6abc.$$

Establish when they are equalities.

In fact, we do not need to restrict ourselves with positive αn requiring only the first of the two conditions (11) but not (12).

Example 13. *For example, $(\frac{1}{2}, \frac{1}{2}, 0) \prec (2, -1, 0)$ and the inequality*

$$2(\sqrt{xy} + \sqrt{xz} + \sqrt{yz}) \leq \frac{x^2}{y} + \frac{x^2}{z} + \frac{y^2}{x} + \frac{y^2}{z} + \frac{z^2}{x} + \frac{z^2}{y}.$$

holds.

Exercise 18. *Prove this extension of the Muirhead Theorem.*

Americans call Muirhead's Theorem a "Bunching Principle." They also use a different notation that can sometimes be useful.

The notion of majorisation can be extended to sequences of the same length. If $s = (s_1, \ldots, s_n)$ and $t = (t_1, \ldots, t_n)$ are two nonincreasing sequences, we say that s *majorizes* t if $s_1 + \cdots + s_n = t_1 + \cdots + t_n$ and $s_1 + \cdots + s_i \geq t_1 + \cdots + t_i$ for $i = 1, \ldots, n$.

Theorem 14 ("Bunching Principle"). *If s and t are two nonincreasing sequences of nonnegative real numbers such that s majorizes t, then*

$$\sum_{\text{sym}} x_1^{s_1} \cdots x_n^{s_n} \geq \sum_{\text{sym}} x_1^{t_1} \cdots x_n^{t_n},$$

where the sums are taken over all $n!$ permutations of variables.

7 Constraints and Homogenisation

Homogenisation is a very important technique, especially in conjunction with Muirhead Theorem. We illustrate it on the following simple example.

Example 14. *Let a, b, c be positive real numbers such that $abc = 1$. Prove that*

$$a + b + c \leq a^2 + b^2 + c^2.$$

Proof. The inequality is not homogeneous in the sense that different terms have different degrees. However the constraint $abc = 1$ is not homogeneous either and we may use this to homogenise the inequality. We make it homogeneous by multiplying the LHS by $\sqrt[3]{abc}$. We will obtain an equivalent inequality

$$a^{\frac{4}{3}}b^{\frac{1}{3}}c^{\frac{1}{3}} + a^{\frac{1}{3}}b^{\frac{4}{3}}c^{\frac{1}{3}} + a^{\frac{1}{3}}b^{\frac{1}{3}}c^{\frac{4}{3}} \leq a^2 + b^2 + c^2,$$

which follows from Muirhead's inequality

$$a^{\frac{2}{3}}b^{\frac{1}{6}}c^{\frac{1}{6}} + a^{\frac{1}{6}}b^{\frac{2}{3}}c^{\frac{1}{6}} + a^{\frac{1}{6}}b^{\frac{1}{6}}c^{\frac{2}{3}} \leq a + b + c$$

since $(1, 0, 0) \succ (\frac{2}{3}, \frac{1}{6}, \frac{1}{6})$. One can also use Weighted AM-GM to show

$$\frac{2}{3}a^2 + \frac{1}{6}b^2 + \frac{1}{6}c^2 \geq \sqrt[6]{a^8b^2c^2} = a^{\frac{4}{3}}b^{\frac{1}{3}}c^{\frac{1}{3}}$$

and then symmetrise this expression. □

Example 15. *Let a, b, c be non-negative real numbers such that $a + b + c = 1$. Prove that*

$$a^3 + b^3 + c^3 + 6abc \geq \frac{1}{4}.$$

Proof. Multiplying by 4 and homogenising, we obtain

$$4(a^3 + b^3 + c^3) + 24abc \geq (a + b + c)^3.$$

Simplifying we get

$$a^3 + b^3 + c^3 + 6abc \geq a^2(b + c) + b^2(a + c) + c^2(a + b),$$

which follows from (10). □

Exercise 19. *Let a, b, c be real numbers such that $abc = -1$. Show that*

$$a^4 + b^4 + c^4 + 3(a + b + c) \geq \frac{a^2}{b} + \frac{a^2}{c} + \frac{b^2}{a} + \frac{b^2}{c} + \frac{c^2}{a} + \frac{c^2}{b}.$$

8 Symmetric Averages

Two more theorems which have been seldom used in Math Olympiad practice so far.

Let $x_1, \ldots, x_n$ be a sequence of non-negative real numbers. We define the symmetric averages of $x_1, \ldots, x_n$ by

$$d_i = \frac{s_i}{\binom{n}{i}},$$

where s_i's are the coefficients of the polynomial

$$(x - x_1)(x - x_2) \ldots (x - x_n) = s_n x^n + \ldots + s_1 x + s_0.$$

For example, if $n = 4$, then

$$\begin{aligned}
d_1 &= \frac{1}{4}(x_1 + x_2 + x_3 + x_4), \\
d_2 &= \frac{1}{6}(x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4), \\
d_3 &= \frac{1}{4}(x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4), \\
d_4 &= x_1x_2x_3x_4.
\end{aligned}$$

Theorem 15 (Newton). *For all $i = 1, \ldots, n-1$*

$$d_i^2 \geq d_{i-1}d_{i+1}.$$

Example 16. *For example, for $n = 4$ we have $d_2^2 \geq d_1d_3$, i.e.*

$$(x_1x_2+x_1x_3+x_1x_4+x_2x_3+x_2x_4+x_3x_4)^2 \geq \frac{9}{4}(x_1+x_2+x_3+x_4)(x_1x_2x_3+x_1x_2x_4+x_1x_3x_4+x_2x_3x_4).$$

or

$$\left(\sum_{\text{sym}} x_1x_2\right)^2 \geq \frac{9}{4}\sum_{\text{sym}} x_1 \cdot \sum_{\text{sym}} x_1x_2x_3.$$

The latter makes sense only if $n = 4$ is agreed upon and implicitely assumed. Otherwise it is confusing.

Theorem 16 (Maclaurin). $d_1 \geq \sqrt{d_2} \geq \sqrt[3]{d_3} \geq \ldots \geq \sqrt[n]{d_n}$.

Example 17. *For example, for $n = 4$ we have $\sqrt{d_2} \geq \sqrt[3]{d_3}$, or equivalently*

$$\left(\sum_{\text{sym}} x_1x_2\right)^3 \geq \frac{27}{2}\left(\sum_{\text{sym}} x_1x_2x_3\right)^2.$$

Example 18. $d_1 \geq \sqrt[n]{d_n}$ *is the familiar AM-GM inequality. Maclaurin's theorem is a refinement of it.*

9 Hints and Solutions to Exercises

1. Check that if $a_1 < a_n$, then the process of transforming $(a_2, \ldots, a_n, a_1)$ into $(a_1, \ldots, a_n)$ described in the theorem will give at least one strict inequality.

2. Same as in Exercise 1.

3. Assume $a \le b \le c$. Then $a^2 \le b^2 \le c^2$. Applying the LHS of Rearrangement inequality to these two sequences we get

$$a^2b + b^2c + c^2a \le a^3 + b^3 + c^3$$

Also we have $a^4 \le b^4 \le c^4$ and $1/c \le 1/b \le 1/a$. Applying the RHS of Rearrangement inequality we obtain

$$a^3 + b^3 + c^3 \le \frac{a^4}{b} + \frac{b^4}{c} + \frac{c^4}{a}.$$

4. Suppose $a \le b \le c$. As $\ln x$ is increasing function of x it is sufficient to compare logarithms of both sides of the inequality, i.e. to establish that $\ln(a^a b^b c^c) \ge \ln(a^b b^c c^a)$. This can be written as

$$a \ln a + b \ln b + c \ln c \ge b \ln a + c \ln b + a \ln c,$$

which is true by Rearrangement inequality since $\ln a \le \ln b \le \ln c$.

5. Apply Cauchy's inequality to $a_i = x_i$ and $b_i = 1/x_i$. To obtain AM-HM inequality divide both sides by n and the sum of reciprocals.

6. Let us make use of Theorem 6 to prove it. We will apply (3) for

$$x_1^2 = \frac{a}{b+c}, \quad x_2^2 = \frac{b}{c+d}, \quad x_3^2 = \frac{c}{d+a}, \quad x_4^2 = \frac{d}{a+b},$$
$$y_1^2 = a(b+c), \quad y_2^2 = b(c+d), \quad y_3^2 = c(d+a), \quad y_4^2 = d(a+b).$$

We will obtain

$$\frac{a}{b+c} + \frac{b}{c+d} + \frac{c}{d+a} + \frac{d}{a+b} \ge (a+b+c+d)^2 \left(a(b+c) + b(c+d) + c(d+a) + d(a+b)\right)^{-1}$$
$$= (a+b+c+d)^2 (2ac + 2bd + ab + bc + cd + da)^{-1}.$$

To show that the RHS is greater or equal than 2 we need to prove that

$$(a+b+c+d)^2 \ge 2(2ac + 2bd + ab + bc + cd + da),$$

which can be checked by simply expanding brackets.

7. Let us note that
$$\frac{1}{1-\sqrt{a}}=\frac{\sqrt{a}}{1-\sqrt{a}}+1,$$
hence we have to prove that
$$\frac{\sqrt{a}}{1-\sqrt{a}}+\frac{\sqrt{b}}{1-\sqrt{b}}+\frac{\sqrt{c}}{1-\sqrt{c}}+\frac{\sqrt{d}}{1-\sqrt{d}}\geq 4.$$
We will prove a more general inequality: for all non-negative $a_1,\dots,a_n$ such that $\sum_{k=1}^n a_k=1$ it is true that
$$\sum_{k=1}^n \frac{\sqrt{a_k}}{1-\sqrt{a_k}}\geq 4,$$
which coincides with the Mongolian question for $n=4$. Let us use Theorem 6 again and apply (3). We obtain
$$\frac{\sqrt{a_k}}{1-\sqrt{a_k}}\geq\left(\sum_{k=1}^n a_k\right)^2\left(\sum_{k=1}^n a_k\sqrt{a_k}(1-\sqrt{a_k}\right)^{-1}=\left(\sum_{k=1}^n a_k\sqrt{a_k}(1-\sqrt{a_k}\right)^{-1}.$$
By AM-GM $a_k\sqrt{a_k}=2a_k\frac{\sqrt{a_k}}{2}\leq\frac{a_k}{4}+a_k^2$, hence
$$\sum_{k=1}^n a_k\sqrt{a_k}\leq\frac{1}{4}\sum_{k=1}^n a_k+\sum_{k=1}^n a_k^2.$$
This implies
$$\sum_{k=1}^n a_k\sqrt{a_k}(1-\sqrt{a_k})\leq\frac{1}{4},$$
which, in turn, implies the inequality.

8. We will apply (4) of Corollary 3. It implies that for some $z_1,\dots,z_n$ such that $z_1^2+\ldots+z_n^2=1$
$$\sqrt{(x_1+y_1)^2+\ldots+(x_n+y_n)^2}=\max\left(\sum_{k=1}^n(x_k+y_k)z_k\right)^2.$$
Using that $\max(u+v)\leq\max u+\max v$, we obtain
$$\sqrt{(x_1+y_1)^2+\ldots+(x_n+y_n)^2}=\max\left(\sum_{k=1}^n(x_k+y_k)z_k\right)^2\leq$$
$$\max\left(\sum_{k=1}^n x_kz_k\right)^2+\max\left(\sum_{k=1}^n y_kz_k\right)^2=\sqrt{x_1^2+\ldots+x_n^2}+\sqrt{y_1^2+\ldots+y_n^2}.$$

9. For any $y_1, \ldots, y_n$ such that $y_1 y_2 \cdots y_n = 1$ AM-GM implies

$$\sqrt[n]{x_1 x_2 \cdots x_n} = \sqrt[n]{(x_1 y_1)(x_2 y_2) \cdots (x_n y_n)} \leq \sum_{k=1}^{n} \frac{x_i y_i}{n}.$$

This becomes an equality iff $x_1 y_1 = \ldots = x_n y_n$. This can be achieved by taking $y_i = \frac{q}{x_i}$, where $q = \sqrt[n]{x_1 x_2 \cdots x_n}$.

10. Using the theorem from the previous exercise we get for some $z_1, \ldots, z_n$ such that $z_1 z_2 \cdots z_n = 1$

$$\sqrt[n]{(x_1 + y_1)(x_2 + y_2) \cdots (x_n + y_n)} = \min \left(\sum_{k=1}^{n} (x_k + y_k) z_k \right) \geq$$

$$\min \left(\sum_{k=1}^{n} x_k z_k \right) + \min \left(\sum_{k=1}^{n} y_k z_k \right) = \sqrt[n]{x_1 x_2 \cdots x_n} + \sqrt[n]{y_1 y_2 \cdots y_n}.$$

11. Use Chebyshev's inequality (6) for sequences $a_1, \ldots, a_n$ and $-b_1, -b_2, \ldots, -l_n$.

12. We may assume that $x_1 \leq x_2 \leq \ldots \leq x_n$. But then $x_1^5 \leq x_2^5 \leq \ldots \leq x_n^5$ and $1/x_1^2 \geq 1/x_2^2 \geq \ldots \geq 1/x_n^2$ so these two sequences are oppositely directed. Hence Chebyshev's Inequality (7) is applicable and gives the result.

13. Since $y = \ln x$ is an increasing function, it is sufficient to prove the same inequality for logarithms. Taking logs we will obtain

$$\left(\frac{a}{a+b+c} \right) \cdot \ln a + \left(\frac{b}{a+b+c} \right) \cdot \ln b + \left(\frac{c}{a+b+c} \right) \cdot \ln c$$

$$\leq \ln \left(\frac{a}{a+b+c} \cdot a + \frac{b}{a+b+c} \cdot b + \frac{c}{a+b+c} \cdot c \right).$$

This is true. We apply Jensen's inequality for the function $y = \ln x$, which is concave down and

$$\alpha_1 = \frac{a}{a+b+c}, \quad \alpha_2 = \frac{b}{a+b+c}, \quad \alpha_3 = \frac{c}{a+b+c}.$$

(Note that $\alpha_1 + \alpha_2 + \alpha_3 = 1$).

14. Take logs on both sides and use concavity of logarithm down.

15. The inequality can be rewritten as follows:

$$(ab + bc + ca)a^2b^2c^2 \leq a^8 + b^8 + c^8.$$

Weighted AM-GM yields

$$\frac{3}{8}a^8 + \frac{3}{8}b^8 + \frac{2}{8}c^8 \geq a^3b^3c^2.$$

Symmetrising it we obtain the inequality we want.

16. For $r = 1$ Schur's inequality yields

$$a(a-b)(a-c) + b(b-c)(b-a) + c(c-a)(c-b) \geq 0,$$

which can be rewritten as

$$a^3 + b^3 + c^3 + 3abc \geq a^2(b+c) + b^2(a+c) + c^2(a+b).$$

17. For all non-negative x_1, x_2, x_3 by Muirheads theorem

$$x^{(1,0,0)} \geq x^{(\frac{2}{3},\frac{1}{3},0)} \geq x^{(\frac{1}{3},\frac{1}{3},\frac{1}{3})}.$$

Hence

$$\begin{aligned}\frac{1}{3}(x_1 + x_2 + x_3) &\geq \frac{1}{6}\left(\sqrt[3]{x_1^2x_2} + \sqrt[3]{x_1^2x_3} + \sqrt[3]{x_2^2x_1} + \sqrt[3]{x_2^2x_3} + \sqrt[3]{x_3^2x_1} + \sqrt[3]{x_3^2x_2}\right) \\ &\geq \sqrt[3]{x_1x_2x_3}\end{aligned}$$

or, after the substitution $a = \sqrt[3]{x_1}$, $b = \sqrt[3]{x_2}$, $c = \sqrt[3]{x_3}$,

$$2(a^3 + b^3 + c^3) \geq a^2(b+c) + b^2(a+c) + c^2(a+b) \geq 6abc.$$

18. Check that the given proof of Muirhead's theorem works.

19. First we homogenise, obtaining

$$a^4 + b^4 + c^4 + a^3(b+c) + b^3(a+c) + c^3(a+b) - 3abc(a+b+c) \geq 0.$$

We now note that $a + b + c$ is a factor of the RHS and the inequality can be rewritten as

$$(a+b+c)(a^3 + b^3 + c^3 - 3abc) \geq 0.$$

Now we see that for $c = -(a+b)$ the second bracket is zero, hence it is divisible by $a + b + c$. Finally we find that the inequality can be written as

$$(a+b+c)^2(a^2 + b^2 + c^2 - ab - bc - ca) \geq 0$$

in which form it is obvious.

10 Further Exercises

17. Prove that for $a, b > 0$
$$\sqrt[n+1]{ab^n} \le \frac{a+nb}{n+1}.$$

18. For positive a, b, c prove that
$$\left(a^3+b^3+c^3\right)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \ge (a+b+c)^2.$$

19. For positive a, b, c prove that
$$\frac{a+b+c}{abc} \le \frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}.$$

20. Prove the inequality
$$a^2(1+b^2)+b^2(1+c^2)+c^2(1+a^2) \ge 6abc.$$

21. For positive a, b, c prove that
$$a^3(b+c)+b^3(c+a)+c^3(a+b) \ge 2(a^2b^2+b^2c^2+c^2a^2).$$

22. Let $P(x)$ be a polynomial with positive coefficients. Prove that if
$$P\left(\frac{1}{x}\right) \ge \frac{1}{P(x)}$$
holds for $x = 1$, then it holds for all $x > 0$.

23. (IMO 1978) Let $a_1, \ldots, a_n, \ldots$ be a sequence of pairwise distinct positive integers. Prove that for all positive integers n
$$\sum_{k=1}^{n} \frac{a_k}{k^2} \ge \sum_{k=1}^{n} \frac{1}{k}.$$

24. (Lagrange's Identity) Prove that for all positive reals $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$
$$\left(\sum x_i^2\right)\left(\sum y_i^2\right) - \left(\sum x_i y_i\right)^2 = \sum_{i<j}(x_i y_j - x_j y_i)^2.$$

25. Prove that two triangles with sides a, b, c and a_1, b_1, c_1 and semiperimeters p and p_1, respectively, are similar if and only if

$$\sqrt{aa_1} + \sqrt{bb_1} + \sqrt{cc_1} = 2\sqrt{pp_1}.$$

26. (Balkan MO, 84) Prove that, if $a_1, \ldots, a_n$ are positive numbers whose sum is 1, then

$$\frac{a_1}{2-a_1} + \frac{a_2}{2-a_2} + \ldots + \frac{a_n}{2-a_n} \geq \frac{n}{2n-1}.$$

27. Prove that for all positive a, b, c

$$\frac{a^n}{b+c} + \frac{b^n}{c+a} + \frac{c^n}{a+b} \geq \frac{a^{n-1}+b^{n-1}+c^{n-1}}{2}.$$

28. Prove that for any positive numbers a, b, c

$$\frac{a}{b+2c} + \frac{b}{c+2a} + \frac{c}{a+2b} \geq 1.$$

29. Let a, b, c be positive reals such that $a+b+c=1$. Prove that

$$\sqrt{ab+c} + \sqrt{bc+a} + \sqrt{ca+b} \geq 1 + \sqrt{ab} + \sqrt{bc} + \sqrt{ca}.$$

30. (Russian winter Camp, 98) Prove that for any positive numbers a, b, c, d

$$\frac{a}{b+2c+3d} + \frac{b}{c+2d+3a} + \frac{c}{d+2a+3b} + \frac{d}{a+2b+3c} \geq \frac{2}{3}.$$

31. (APMO, 91) Let $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ be positive numbers such that

$$a_1 + a_2 + \cdots + a_n = b_1 + b_2 + \cdots + b_n.$$

Prove that

$$\frac{a_1^2}{a_1+b_1} + \frac{a_2^2}{a_2+b_2} + \ldots + \frac{a_n^2}{a_n+b_n} \geq \frac{a_1+a_2+\ldots+a_n}{2}.$$